Non-Linear Problems

Almost Linear Systems

Consider the system of equations:

(1) $Y' = AY + g(Y)$

(here Y is a matrix).

Suppose that the origin is an ***isolated*** equilibrium point (that is, there is some circle around the origin where the origin is the only equlibrium point) for this system of equations. Furthermore, let's assume that $\det A \neq 0$. This ensures that the origin is also an isolated critical point of the linear system

$$Y' = AY.$$

Also, let's assume that the components of g have continuous partial derivatives and are small near the origin:

$$\frac{\|g(Y)\|}{\|Y\|} \to 0 \quad \text{as } Y \to 0.$$

With these conditions, the system (1) is called an ***almost linear system*** in the neighborhood of the equlibrium point $(0, 0)$.

■ Example

Consider the following system:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^2 - x\,y \\ -0.75\,x\,y - 0.25\,y^2 \end{pmatrix}.$$

Note that the origin is an equilbrium point and that the system is in the form (1). Also, it is clear that $\det A \neq 0$.

It is also easy to check that the origin is an isolated equilibrium point. The equilibrium points satisfy the equations:

$$x' = x - x^2 - x\,y = 0,$$

$$y' = 0.5\,y - 0.75\,x\,y - 0.25\,y^2 = 0.$$

In[54]:=
```
Solve[{x - x^2 - x y == 0, 0.5 y - 0.75 x y - 0.25 y^2 == 0},
  {x, y}]
```

Out[54]=
```
{{x → 0., y → 0.}, {x → 0., y → 2.},
  {x → 0.5, y → 0.5}, {x → 1., y → 0.}}
```

So the origin is an isolated equilibrium point.

In[55]:=
```
f[x_, y_] := -x^2 - x y
```

In[56]:=
```
g[x_, y_] := -0.75 x y - 0.25 y^2
```

In[57]:=
```
normy[x_, y_] := Sqrt[f[x, y]^2 + g[x, y]^2]
```

In[58]:=
```
normy[x, y]
```

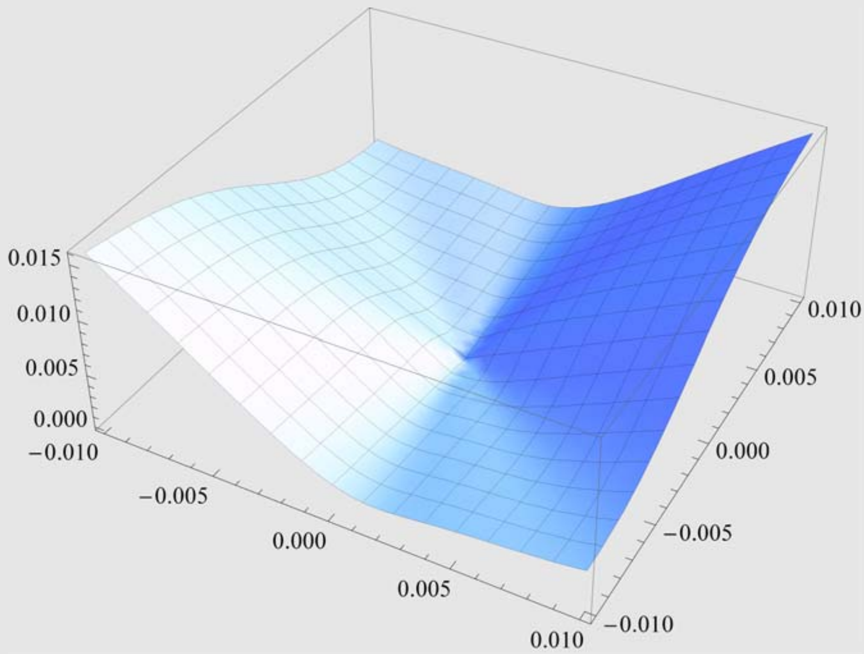
Out[58]=

$$\sqrt{\left(-x^2 - x\,y\right)^2 + \left(-0.75\,x\,y - 0.25\,y^2\right)^2}$$

In[59]:=

```
nplot = Plot3D[normy[x, y] / Sqrt[x^2 + y^2],
    {x, -0.01, 0.01}, {y, -0.01, .01}]
```

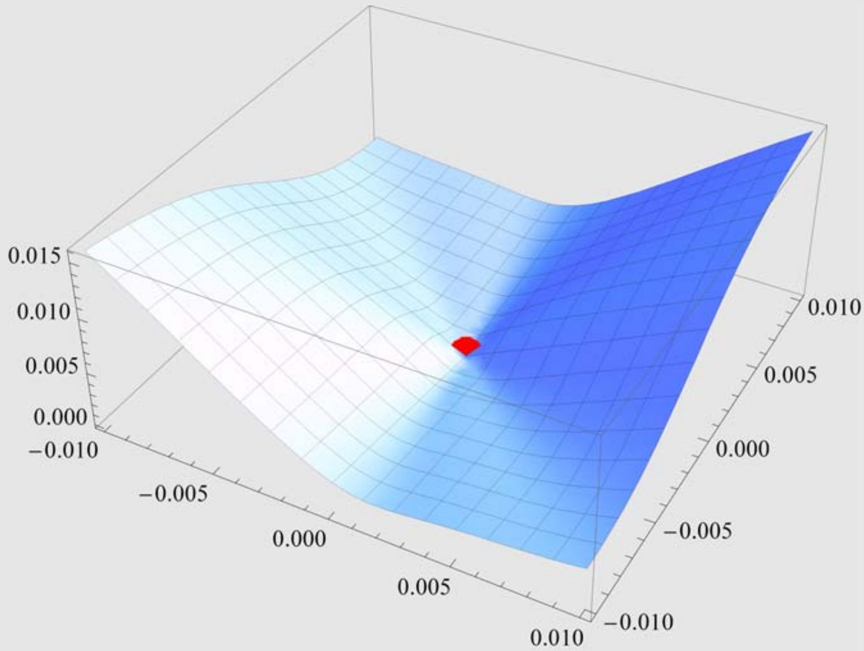
Out[59]=



In[60]:=

```
Show[nplot,
  Graphics3D[{PointSize[0.05], Hue[1],
      Point[{0, 0, 0}]}]]
```

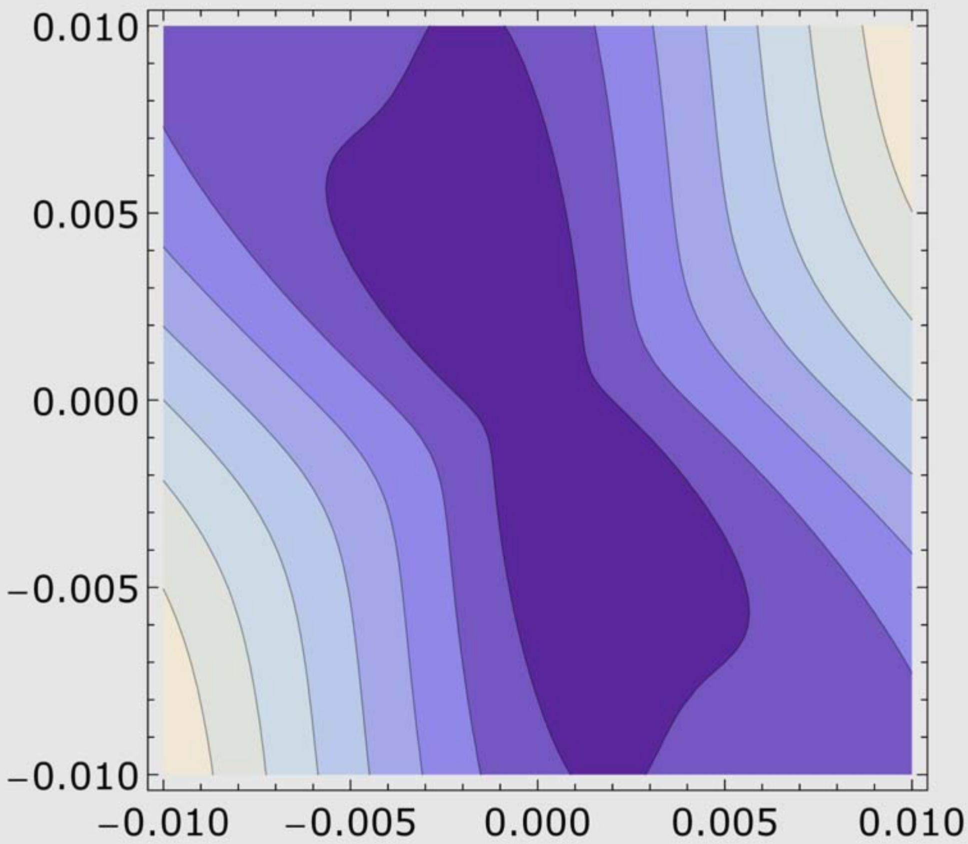
Out[60]=



In[61]:=

```
ContourPlot[normy[x, y] / Sqrt[x^2 + y^2], {x, -.01, .01},
  {y, -.01, .01}]
```

Out[61]=



It is clear that

$$\frac{\|g(Y)\|}{\|Y\|} \to 0 \quad \text{as } Y \to 0.$$

■ Theorem

(Loosely Stated!)

The type and stability of an equlibrium point of the linearized system (2) is the same for the almost linear system (1).

Pendulum Problems

Undamped Pendulum

An undamped pendulum of length *l* is governed by the differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

Let $x = \theta$ and $y = \frac{d\theta}{dt}$ and we can rewrite the second order DE as the system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\left(\frac{g}{l}\right)\sin x$$

We can rewrite this system using the fact that $\sin x = x + (\sin x - x)$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -g/l & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{g}{l}\begin{pmatrix} 0 \\ \sin x - x \end{pmatrix}.$$

It's not difficult to see that this is an almost linear system.

■

Damped Pendulum

Let's assume that there is a damping force proportional to the angu lar speed, that is, there is a damping force $c \left| \frac{d\theta}{dt} \right|$. The principle of angular momentum then gives the following equation:

$$m\, l^2 \frac{d^2\theta}{dt^2} = -c\, l \frac{d\theta}{dt} - m\, g\, l \sin\theta$$

or,

$$\frac{d^2\theta}{dt^2} \frac{c}{m\, l}\, d\theta + \frac{g}{l} \sin\theta = 0.$$

Again, let $x = \theta$ and $y = \frac{\mathrm{d}\theta}{\mathrm{dt}}$and we get the system

$$\frac{\mathrm{dx}}{\mathrm{dt}} = y$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = -\frac{g}{l} \sin x - \frac{c}{\mathrm{ml}}\, y$$

The origin is an isolated critical point of this system. Intuitively, we expect the origin the asymptotically stable. To see this, let's rewrite the system as an almost linear system.

$$\frac{\mathrm{dx}}{\mathrm{dt}} = y$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = -\frac{g}{l}\, x - \frac{c}{\mathrm{ml}}\, y - \frac{g}{l} (\sin x - x)$$

In[62]:=
```
<< "VectorFieldPlots`"
```

In[63]:=
```
doit[l_, g_, m_, c_, x0_, y0_] :=
  Module[{vfield, solution, phase},
    vfield = (Needs["VectorFieldPlots`"];
        VectorFieldPlots`VectorFieldPlot[
          {y, -(g x)/l - (c y)/(m l) - (g (Sin[x] - x))/l}, {x, -3, 3},
          {y, -3, 3}, ScaleFunction -> (1 &),
          Frame -> True, Axes -> True]);
    solution =
      NDSolve[
        {x'[t] == y[t],
          y'[t] == -(g x[t])/l - (c y[t])/(m l) - (g (Sin[x[t]] - x[t]))/l,
          x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 20}];
    phase = ParametricPlot[{x[t], y[t]} /. First[solution],
        {t, 0, 20},
        PlotStyle -> {Thickness[0.01`],
            RGBColor[1, 0, 0]}, Compiled -> False];
    Show[vfield, phase]]
```

In[64]:=
```
doit[1, 9.8, 1, 2, 1, 2]
```

NDSolve::dsvar : 0 cannot be used as a variable. ≫

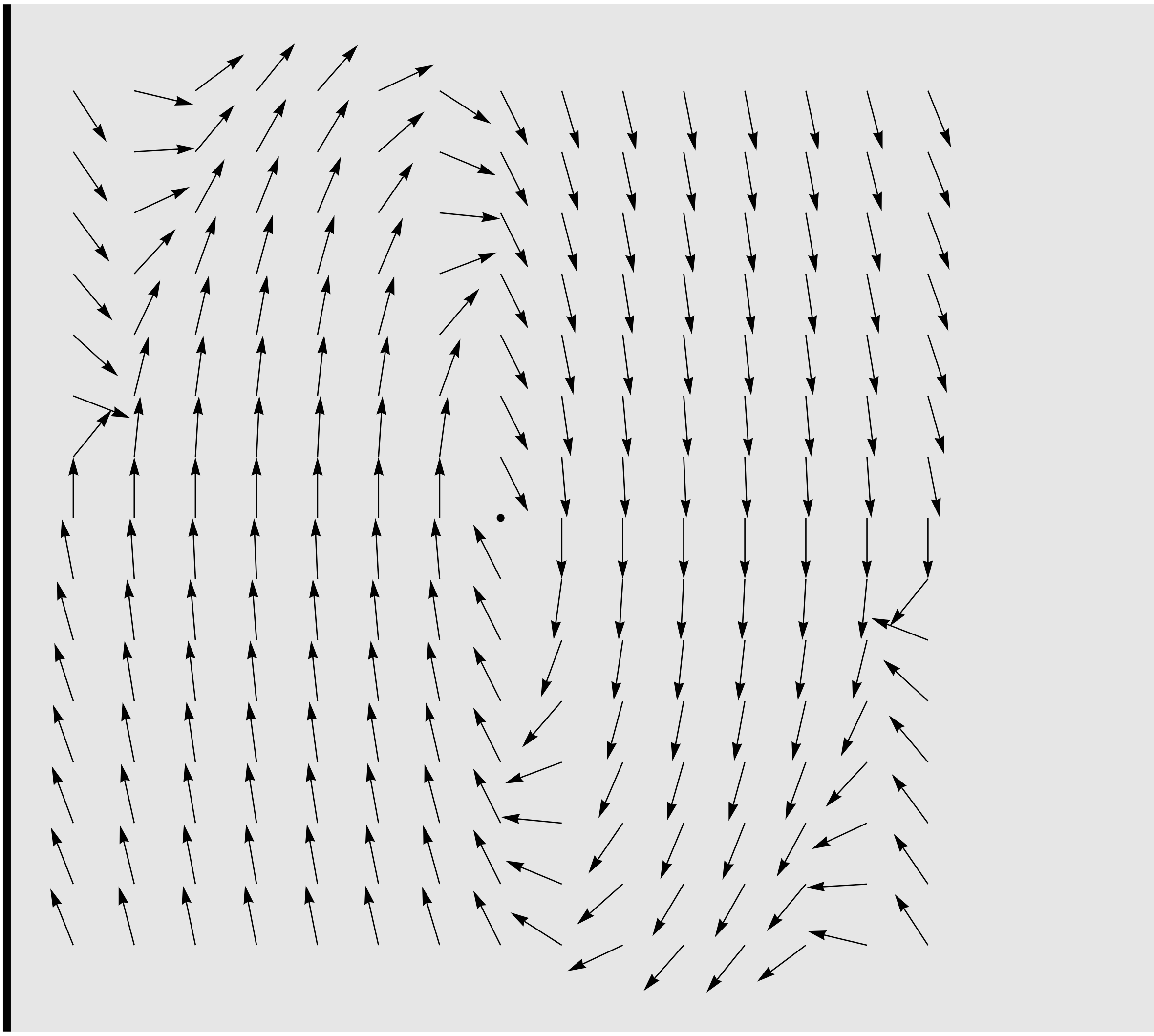
ReplaceAll::reps :
{x′[0] == y[0], y′[0] == −9.8 (Sin[x[0]] − x[0]) − 9.8 x[0] − 2 y[0], x[0] == 1, y[0] == 2}
is neither a list of replacement rules nor a valid
dispatch table, and so cannot be used for replacing. ≫

ReplaceAll::reps :
{x′[0.] == y[0.], y′[0.] == −9.8 (Sin[x[0.]] − 1. x[0.]) − 9.8 x[0.] − 2. y[0.], x[0.] == 1., y[0.] ==
2.} is neither a list of replacement rules nor
a valid dispatch table, and so cannot be used for replacing. ≫

ReplaceAll::reps :
{x′[0] == y[0], y′[0] == −9.8 (Sin[x[0]] − x[0]) − 9.8 x[0] − 2 y[0], x[0] == 1, y[0] == 2}
is neither a list of replacement rules nor a valid
dispatch table, and so cannot be used for replacing. ≫

General::stop :
Further output of ReplaceAll::reps will be suppressed during this calculation. ≫

Out[64]=



Look at the linearized system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\frac{g}{l} x - \frac{c}{ml} y$$

What are the eigenvalues?

In[65]:=
```
Eigensystem[{{0, 1}, {-g/l, -c/(m l)}}]
```

Out[65]=
$$\left\{\left\{\frac{-c-\sqrt{c^2-4\,g\,l\,m^2}}{2\,l\,m}, \frac{-c+\sqrt{c^2-4\,g\,l\,m^2}}{2\,l\,m}\right\}, \left\{\left\{-\frac{c-\sqrt{c^2-4\,g\,l\,m^2}}{2\,g\,m}, 1\right\}, \left\{-\frac{c+\sqrt{c^2-4\,g\,l\,m^2}}{2\,g\,m}, 1\right\}\right\}\right\}$$

If $c^2 - 4\,g\,l\,m^2 > 0$, then there are two distinct real, negative eigenvalues and the origin is an asymptotically stable sink.

If $c^2 - 4\,g\,l\,m^2 = 0$, then there is a repeated negative real eigenvalue. The origin will be asymptotically stable, either a spiral sink or a center.

If $c^2 - 4\,g\,l\,m^2 < 0$, then the eigenvalues are complex with negative real part. The origin is a asymptotically stable spiral point.

Thus, for small damping, the origin is a stable spiral point of the almost linear system.

But the almost linear system has other equilibrium points. The other points are at

$$x = n\pi, \quad y = 0, \quad n = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

which correspond to

$$\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots, \qquad \frac{d\theta}{dt} = 0$$

Intuitively, we expect that the points corresponding to $\theta = \pm 2\pi, \pm 4\pi, \ldots$ are asymptotically stable spiral points, while the points corresponding to $\theta = \pm\pi$, $\pm 3\pi$, ... are unstable saddle points. How can we see this?

In[66]:=
```
doit2[l_, g_, m_, c_, x0_, y0_] :=
  Module[{vfield, solution, phase},
    vfield = (Needs["VectorFieldPlots`"];
        VectorFieldPlots`VectorFieldPlot[
          {y, -g x/l - c y/(m l) - g (Sin[x] - x)/l}, {x, π, 4 π},
          {y, -5, 5}, ScaleFunction → (1 &),
          Frame → True, Axes → True]);
    solution =
      NDSolve[
        {x'[t] == y[t],
          y'[t] == -g x[t]/l - c y[t]/(m l) - g (Sin[x[t]] - x[t])/l,
          x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 20}];
    phase = ParametricPlot[{x[t], y[t]} /. First[solution],
        {t, 0, 20},
        PlotStyle → {Thickness[0.01`],
            RGBColor[1, 0, 0]}, Compiled → False];
    Show[vfield, phase]]
```

In[67]:=
```
doit2[1, 9.8, 1, 2, 4, 2]
```

NDSolve::dsvar : 0 cannot be used as a variable. ≫

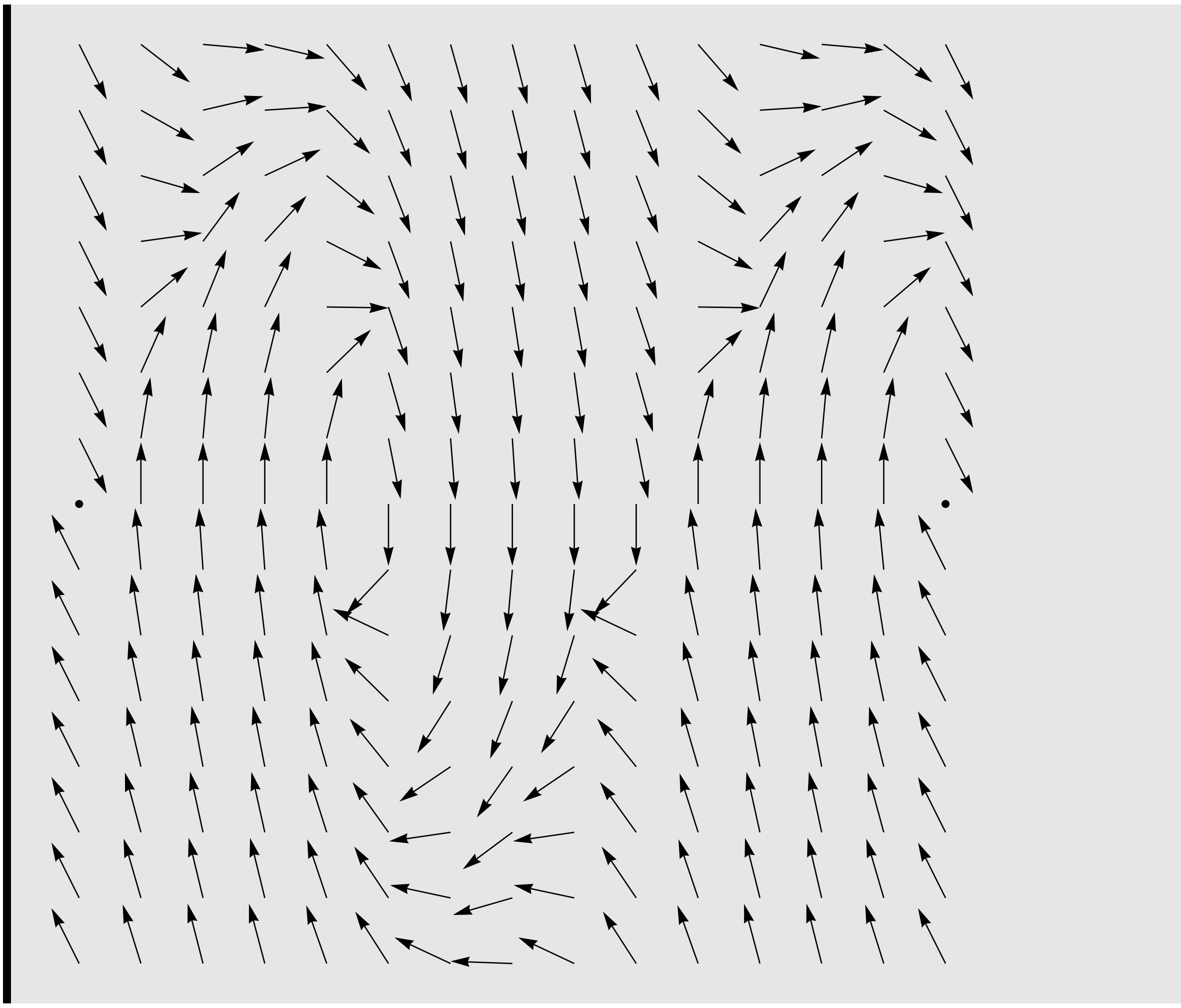
ReplaceAll::reps :
{x′[0] == y[0], y′[0] == −9.8 (Sin[x[0]] – x[0]) – 9.8 x[0] – 2 y[0], x[0] == 4, y[0] == 2}
is neither a list of replacement rules nor a valid
dispatch table, and so cannot be used for replacing. ≫

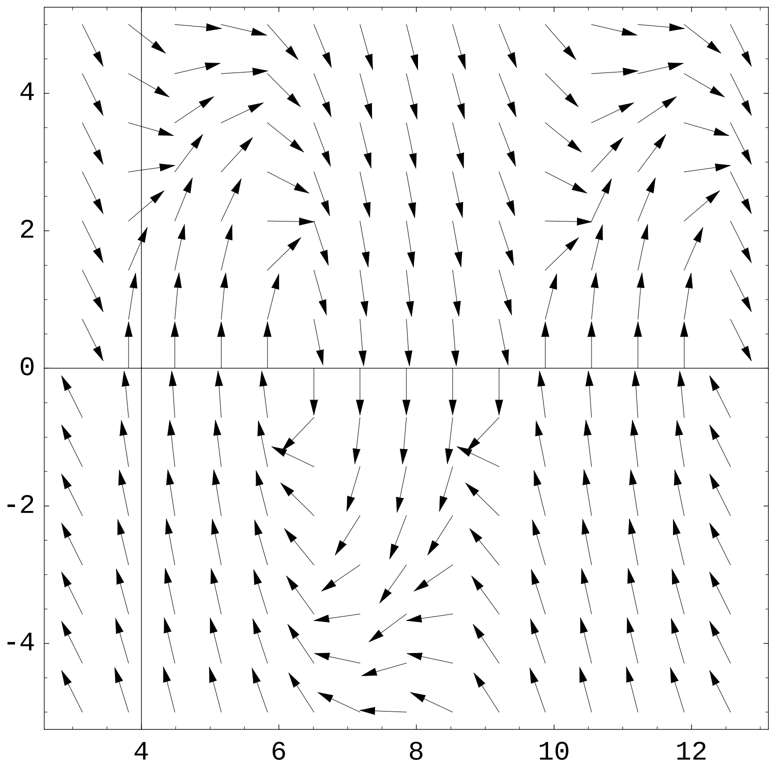
ReplaceAll::reps :
{x′[0.] == y[0.], y′[0.] == −9.8 (Sin[x[0.]] – 1. x[0.]) – 9.8 x[0.] – 2. y[0.], x[0.] == 4., y[0.] ==
2.} is neither a list of replacement rules nor
a valid dispatch table, and so cannot be used for replacing. ≫

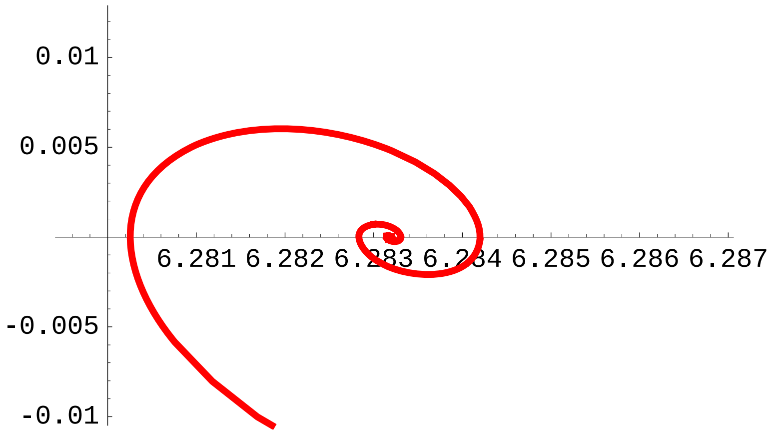
ReplaceAll::reps :
{x′[0] == y[0], y′[0] == −9.8 (Sin[x[0]] – x[0]) – 9.8 x[0] – 2 y[0], x[0] == 4, y[0] == 2}
is neither a list of replacement rules nor a valid
dispatch table, and so cannot be used for replacing. ≫

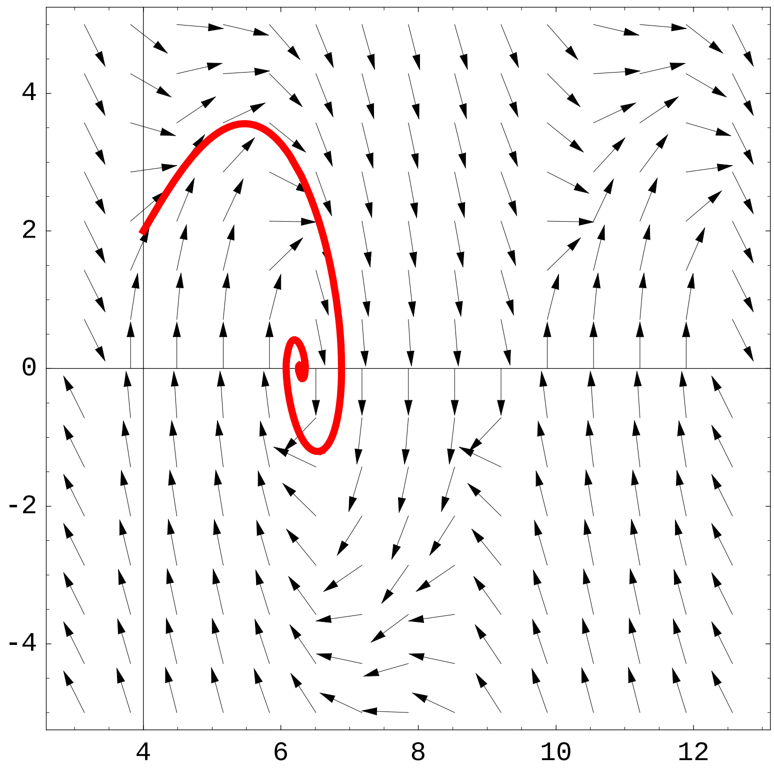
General::stop :
Further output of ReplaceAll::reps will be suppressed during this calculation. ≫

Out[67]=




4
2
0
-2
-4
4
6
8
10
12


0.01
0.005
-0.005
-0.01
6.281 6.282 6.283 6.284 6.285 6.286 6.287



- Graphics -

You can see that the vector field is actually periodic.

Now, let's look close to the point corresponding to $\theta = \pi$.

In[68]:=
```
doit3[l_, g_, m_, c_, x0_, y0_] :=
  Module[{vfield, solution, phase},
    vfield = (Needs["VectorFieldPlots`"];
        VectorFieldPlots`VectorFieldPlot[
          {y, -(g x)/l - (c y)/(m l) - (g (Sin[x] - x))/l}, {x, 0, 2 π},
          {y, -2, 2}, ScaleFunction -> (1 &),
          Frame -> True, Axes -> True]);
    solution =
      NDSolve[
        {x'[t] == y[t],
          y'[t] == -(g x[t])/l - (c y[t])/(m l) - (g (Sin[x[t]] - x[t]))/l,
          x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 20}];
    phase = ParametricPlot[{x[t], y[t]} /. First[solution],
        {t, 0, 2},
        PlotStyle -> {Thickness[0.01`],
            RGBColor[1, 0, 0]}, Compiled -> False];
    Show[vfield, phase]]
```

In[69]:=

```
doit3[1, 9.8, 1, 1, 3.15, 0]
```

NDSolve::dsvar : 0 cannot be used as a variable. ≫

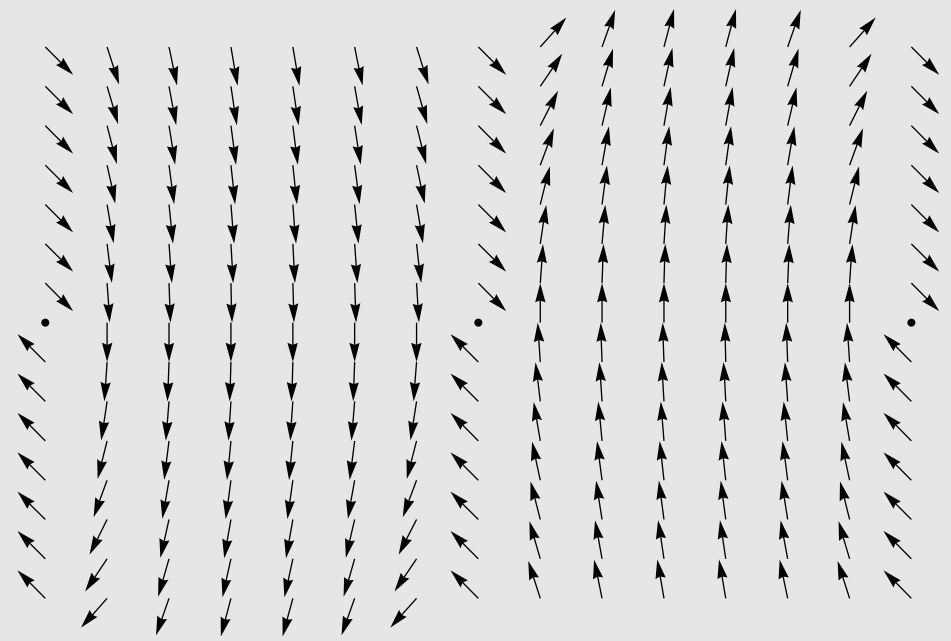
ReplaceAll::reps :
{x′[0] == y[0], y′[0] == −9.8 (Sin[x[0]] − x[0]) − 9.8 x[0] − y[0], x[0] == 3.15, y[0] == 0}
is neither a list of replacement rules nor a valid
dispatch table, and so cannot be used for replacing. ≫

ReplaceAll::reps :
{x′[0.] == y[0.], y′[0.] == −9.8 (Sin[x[0.]] − 1. x[0.]) − 9.8 x[0.] − 1. y[0.], x[0.] == 3.15, y[0.] == 0.} is neither a list of replacement rules
nor a valid dispatch table, and so cannot be used for replacing. ≫

ReplaceAll::reps :
{x′[0] == y[0], y′[0] == −9.8 (Sin[x[0]] − x[0]) − 9.8 x[0] − y[0], x[0] == 3.15, y[0] == 0}
is neither a list of replacement rules nor a valid
dispatch table, and so cannot be used for replacing. ≫

General::stop :
Further output of ReplaceAll::reps will be suppressed during this calculation. ≫

Out[69]=



Notice that the trajectory changes completely depending on which side of the point $(\pi, 0)$ we start at.

From a more analytical sense, we can analyze the behavior of these other equlibrium points by performing a translation our problem by a change of variables to move the point in question, $(\pi, 0)$, to the origin.

Let,

$$x = \pi + u, \quad y = 0 + v.$$

Subsituting back into the original almost linear system produces

$$\frac{du}{dt} = v$$
$$\frac{dv}{dt} = -\frac{c}{ml} v + \frac{g}{l} \sin u$$

Now, $(u, v) = (0, 0)$ is a critical point of this system. Let's rewrite it as an almost linear system

$$\frac{du}{dt} = v$$
$$\frac{dv}{dt} = \frac{g}{l} u - \frac{c}{ml} v + \frac{g}{l} (\sin u - u)$$

It is to show that the $(0, 0)$ is a saddle point for the linearized system.

Rezolvarea sistemelor de ecuatii diferentiale

In[70]:=
```
Clear["Global`*"];
```

In[71]:=
```
ClearAll[x, y]
```

Fie sistemul:

x'[t]-2y[t]=0

y'[t]+x[t]+y[t]=0

cu conditiile initiale: x[0]=1; y[0]=2

In[72]:=
```
sol = DSolve[{D[x[t], t] == 2 * y[t], D[y[t], t] == -x[t] - y[t],
   x[0] == 1, y[0] == 2}, {x[t], y[t]}, t]
```

Out[72]=

$$\left\{\left\{x[t] \to \frac{1}{7} e^{-t/2} \left(7 \mathrm{Cos}\left[\frac{\sqrt{7}\, t}{2}\right] + 9 \sqrt{7}\, \mathrm{Sin}\left[\frac{\sqrt{7}\, t}{2}\right]\right), y[t] \to -\frac{2}{7} e^{-t/2} \left(-7 \mathrm{Cos}\left[\frac{\sqrt{7}\, t}{2}\right] + 2 \sqrt{7}\, \mathrm{Sin}\left[\frac{\sqrt{7}\, t}{2}\right]\right)\right\}\right\}$$

In[73]:=
```
sol1[t_] = First[x[t] /. sol]
sol2[t_] = First[y[t] /. sol]
```

Out[73]=

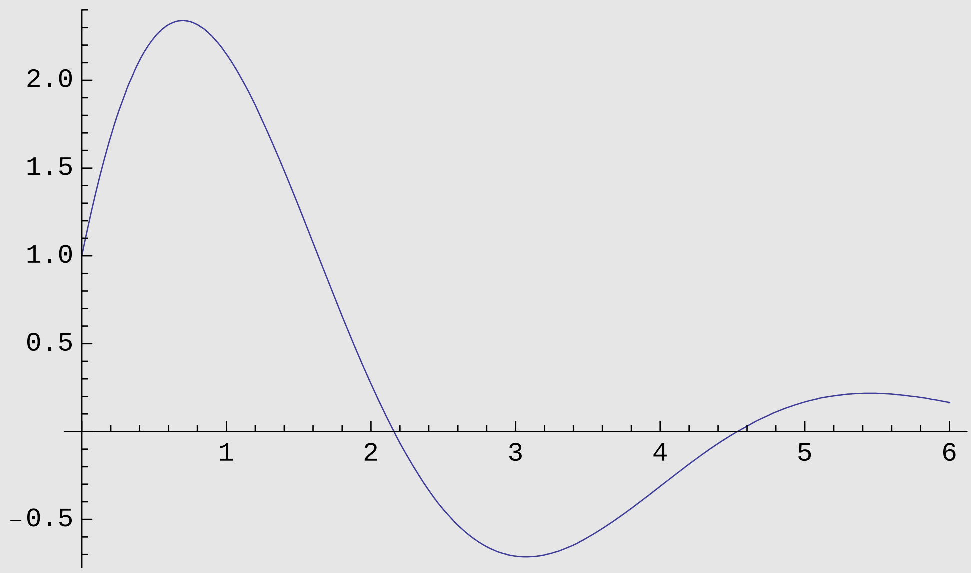
$$\frac{1}{7} e^{-t/2} \left(7 \mathrm{Cos}\left[\frac{\sqrt{7}\, t}{2}\right] + 9 \sqrt{7}\, \mathrm{Sin}\left[\frac{\sqrt{7}\, t}{2}\right]\right)$$

Out[74]=

$$-\frac{2}{7} e^{-t/2} \left(-7 \mathrm{Cos}\left[\frac{\sqrt{7}\, t}{2}\right] + 2 \sqrt{7}\, \mathrm{Sin}\left[\frac{\sqrt{7}\, t}{2}\right]\right)$$

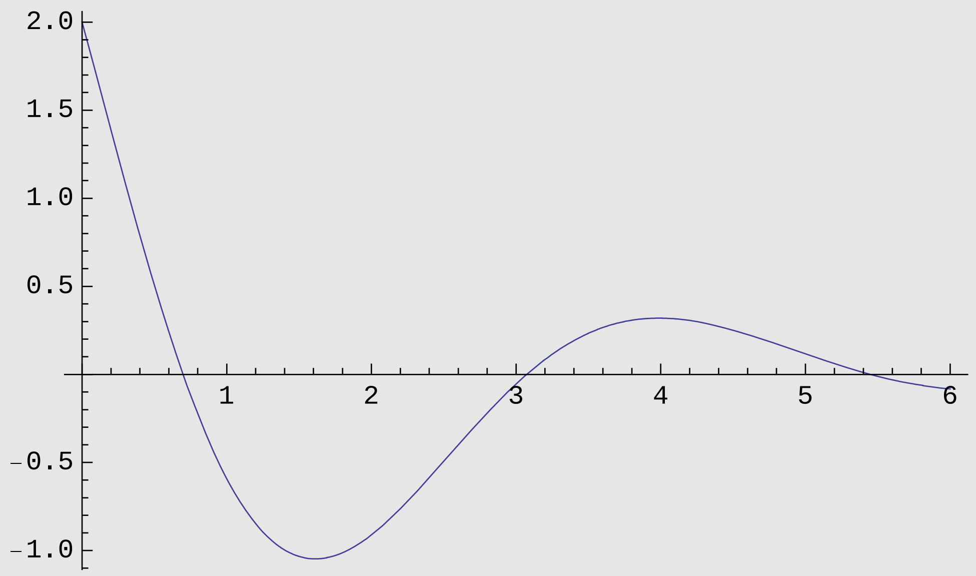
In[75]:=
```
Plot[sol1[t], {t, 0, 6}]
```

Out[75]=



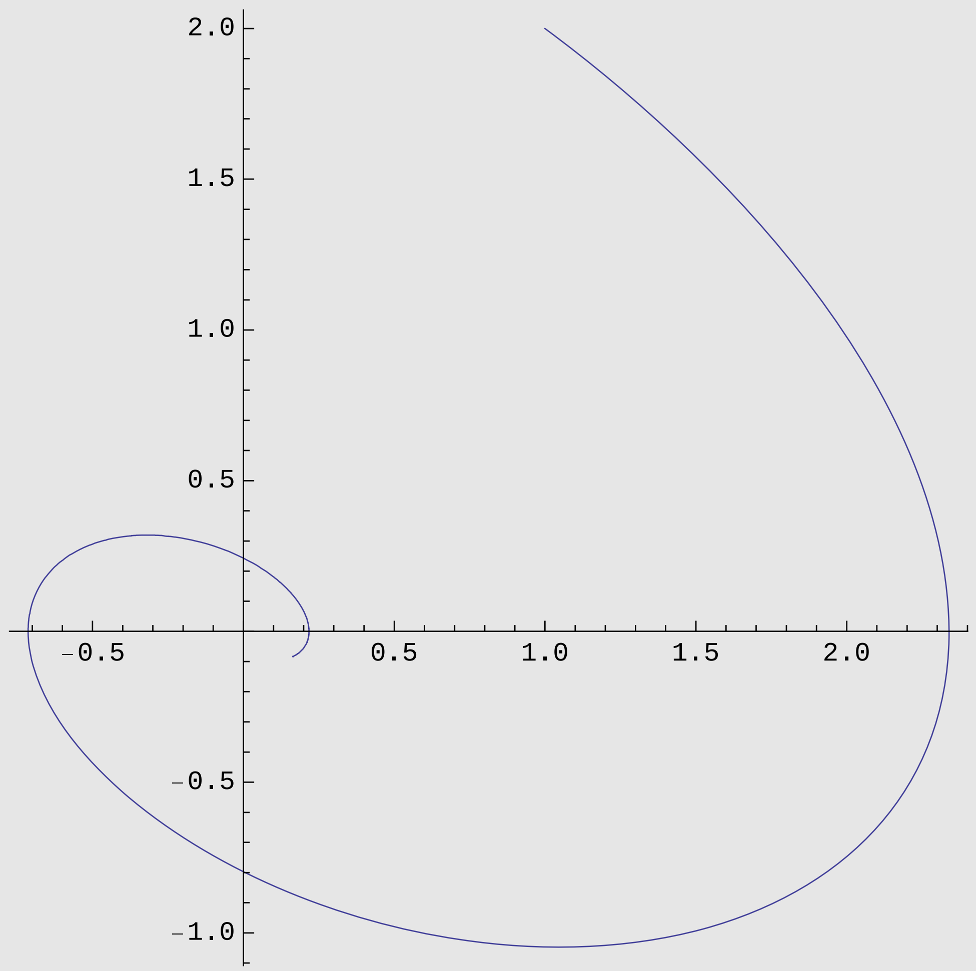
In[76]:=
```
Plot[sol2[t], {t, 0, 6}]
```

Out[76]=



In[77]:=
```
para1 = ParametricPlot[{sol1[t], sol2[t]}, {t, 0, 6}]
```

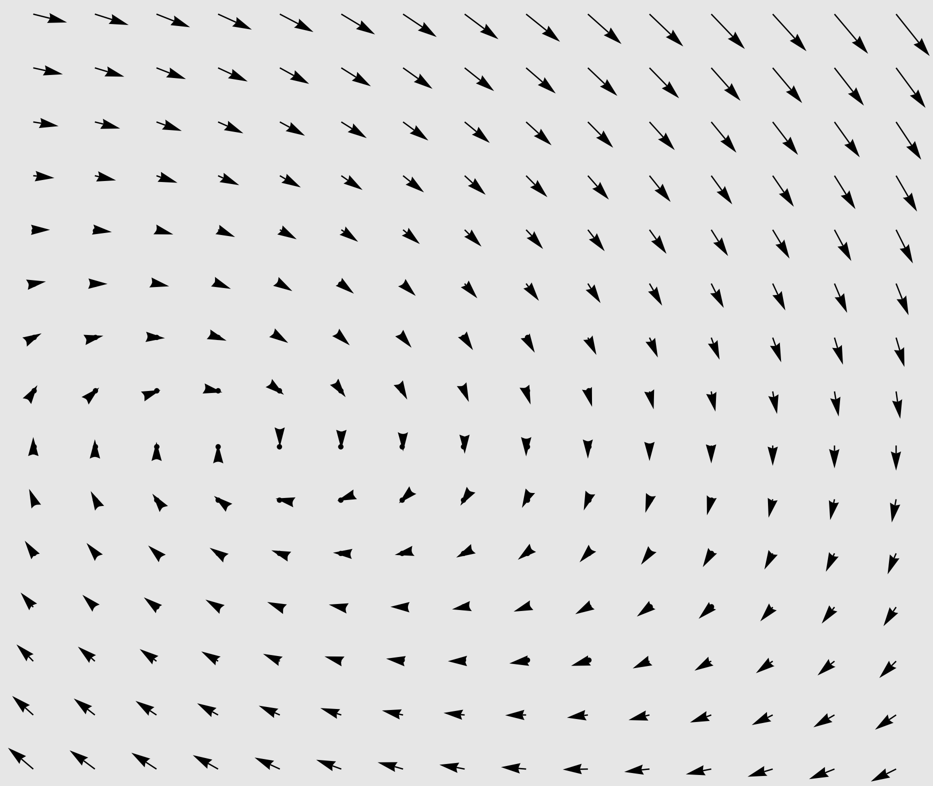
Out[77]=



In[78]:=
```
Needs["VectorFieldPlots`"]
```

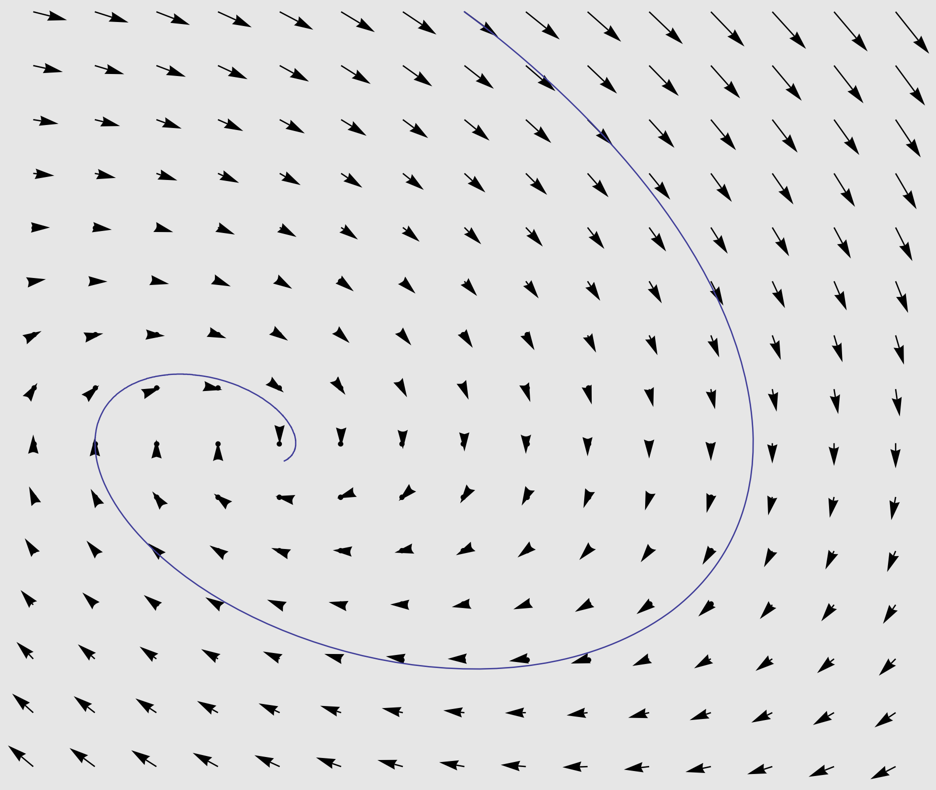
In[79]:=
```
vectorplt = (Needs["VectorFieldPlots`"];
  VectorFieldPlots`VectorFieldPlot[{2 y, -x - y}, {x, -1, 3},
   {y, -1.5`, 2}])
```

Out[79]=



In[80]:=
```
Show[vectorplt, para1]
```

Out[80]=



Fie sistemul

x'[t]+x[t]-2y[t]-1=0

y'[t]-x[t]-3z[t]-1=0

z'[t]-2y[t]+z[t]=0

cu conditiile initiale: x[0]=6; y[0]=2; z[0]=4

In[81]:=
```
soln =
 Simplify[
  DSolve[{D[x[t], t] == -x[t] + 2 * y[t] + t,
    D[y[t], t] == x[t] + 3 * y[t] - z[t] + 1, D[z[t], t] == 2 * y[t] - z[t],
    x[0] == 6, y[0] == 2, z[0] == 4}, {x[t], y[t], z[t]}, t]]
```

Out[81]=

$$\left\{\left\{x[t] \to \frac{1}{72} e^{-t} \left(369 + 103 e^{4t} - 108 t + 8 e^{t} (-5 + 3 t)\right),\right.\right.$$

$$y[t] \to \frac{1}{36} \left(-4 - 27 e^{-t} + 103 e^{3t} - 12 t\right),$$

$$\left.\left.z[t] \to \frac{1}{72} e^{-t} \left(153 + 103 e^{4t} + e^{t} (32 - 48 t) - 108 t\right)\right\}\right\}$$

In[82]:=
```
sol1[t_] = First[x[t] /. soln]
sol2[t_] = First[y[t] /. soln]
sol3[t_] = First[z[t] /. soln]
```

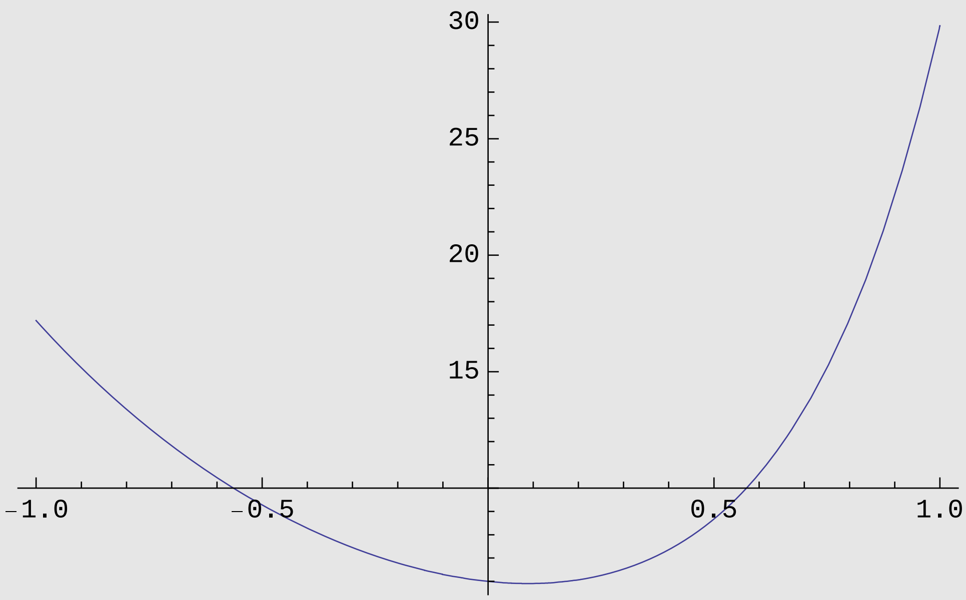
Out[82]= $\frac{1}{72} e^{-t} \left(369 + 103 e^{4t} - 108 t + 8 e^{t} (-5 + 3 t)\right)$

Out[83]= $\frac{1}{36} \left(-4 - 27 e^{-t} + 103 e^{3t} - 12 t\right)$

Out[84]= $\frac{1}{72} e^{-t} \left(153 + 103 e^{4t} + e^{t} (32 - 48 t) - 108 t\right)$

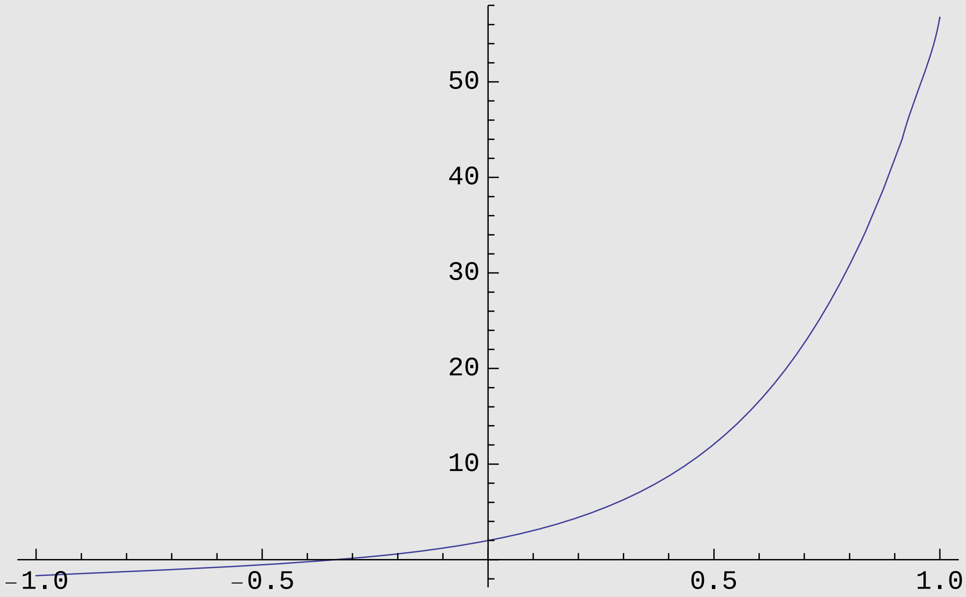
In[85]:=
```
Plot[sol1[t], {t, -1, 1}]
```

Out[85]=



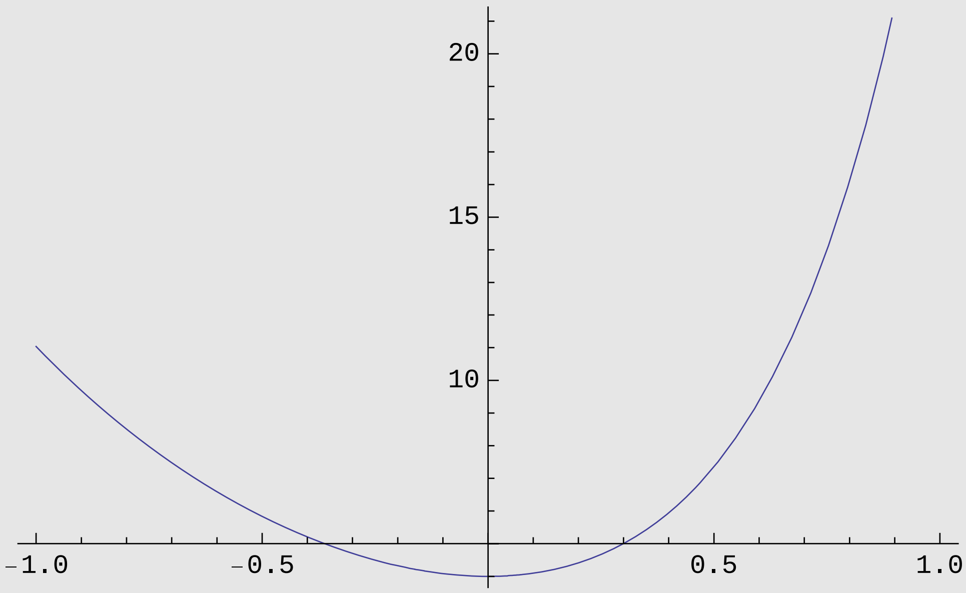
In[86]:=
```
Plot[sol2[t], {t, -1, 1}]
```

Out[86]=



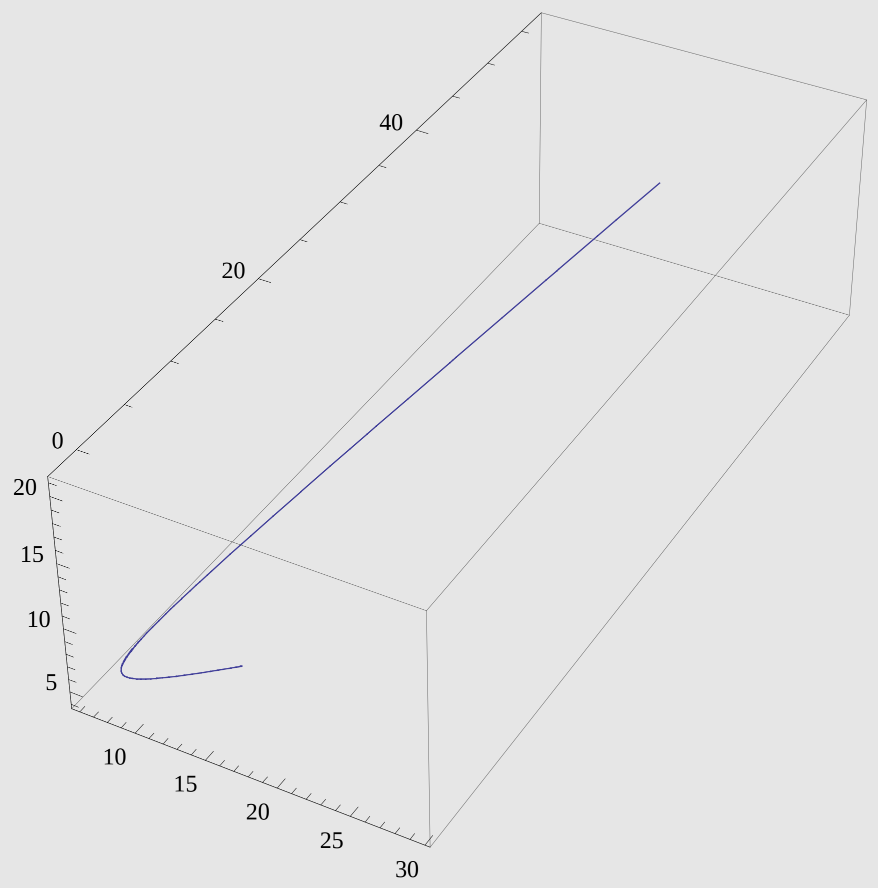
In[87]:=
```
Plot[sol3[t], {t, -1, 1}]
```

Out[87]=



In[88]:=
```
plot3d = ParametricPlot3D[{sol1[t], sol2[t], sol3[t]}, {t, -1, 1}]
```

Out[88]=



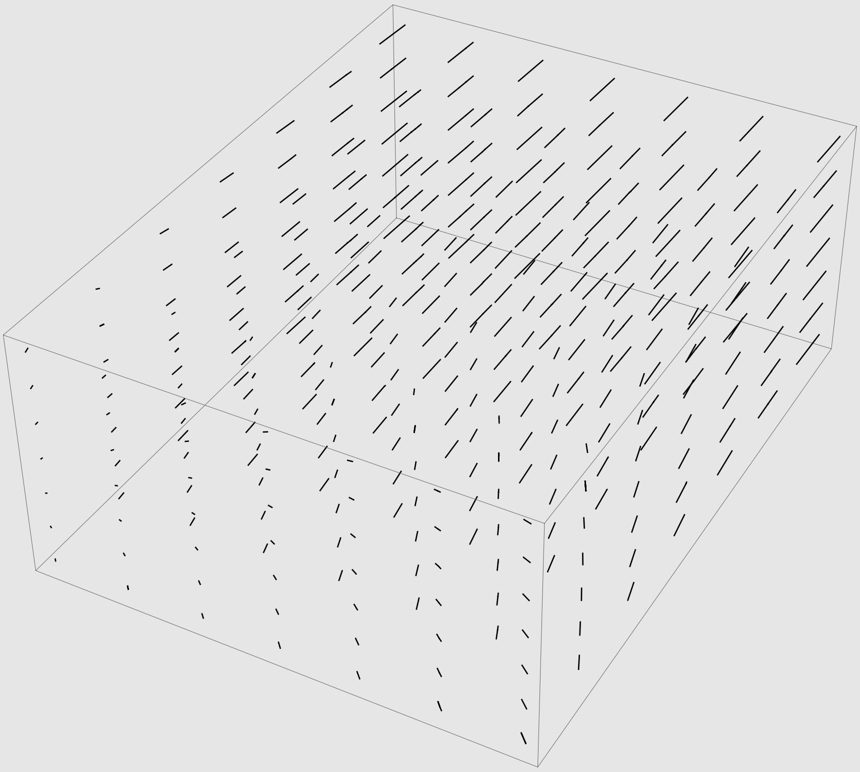
In[89]:=
```
Needs["VectorFieldPlots`"]
```

In[90]:=
```
t = 0
```

Out[90]=
```
0
```

In[91]:=
```
vplt3d = VectorFieldPlot3D[{-x + 2 y + t, x + 3 y - z + 1, 2 y - z},
  {x, 10, 50}, {y, 0, 50}, {z, 0, 20}]
```

Out[91]=



In[92]:=
```
Show[plot3d, vplt3d]
```

Out[92]=